International Journal of Theoretical Physics, Vol. 42, No. 12, December 2003 (© 2003)

Second-Order Linear Differential Equations for Effective Barotropic FRW Cosmologies

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Received February 17, 2003

In 1999, Faraoni wrote a simple second-order linear differential equation for FRW cosmologies with barotropic fluids. His results have been extended by Rosu, who employed techniques belonging to nonrelativistic supersymmetry to obtain time-dependent effective adiabatic indices. Further extensions are presented here using the known connection between the linear second-order differential equations and Dirac-like equations in the same supersymmetric context. These extensions are equivalent to adding an imaginary part to the effective adiabatic index, which is proportional to the mass parameter of the Dirac spinor. The natural physical interpretation of the imaginary part is related to the particular dissipation and instabilities of the effective barotropic FRW hydrodynamics that are introduced by means of this supersymmetric scheme.

KEY WORDS: cosmology; supersymmetry; barotropic fluids; Dirac equations; dissipation.

1. INTRODUCTION

The barotropic FRW cosmologies obey the following set of differential equations:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p), \tag{1}$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{\kappa}{a^2}, \tag{2}$$

$$p = (\gamma - 1)\rho, \tag{3}$$

where a is the scale factor of the universe, ρ and p are the energy density and the pressure, respectively, of the perfect fluid of which a classical universe is usually

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0020-7748/03/1200-2923/0

assumed to be made of, $\kappa = 0, \pm 1$ is the curvature index of the flat, closed, open universe, respectively, and γ is the constant adiabatic index of the cosmological fluid. Recently, Faraoni (1999) proposed the "Riccati route" of solving the system of Eqs. (1)–(3) and Rosu used Faraoni's approach to introduce a supersymmetric class of cosmological fluids possessing time-dependent adiabatic indices (Rosu, 2000; Nowakowski and Rosu, 2002). It was claimed that these fluids can provide a simple explanation for a currently accelerating universe (Perlmutter *et al.*, 1997, 1999).

In this work, we review the supersymmetric factorization methods for barotropic FRW cosmologies in Section 2. Next, in Section 3, we present corresponding Dirac-like (first-order) coupled differential equations and their associated second-order differential equations and discuss them in a formal way. In particular, the bosonic and fermionic partner solutions can be written as the components of a Dirac spinor of zero mass, see 3.1, whereas in 3.2 and 3.3 we do minimal extensions of this known result. We end up the work with a short conclusion section.

2. SUPERSYMMETRIC (FACTORIZATION) METHODS

Combining Eqs. (1)–(3) and using the conformal time variable η defined by $dt = a(\eta)d\eta$ one gets the equation

$$\frac{a''}{a} + (c-1)\left(\frac{a'}{a}\right)^2 + c\kappa = 0. \tag{4}$$

where $c = \frac{3}{2}\gamma - 1$ The case $\kappa = 0$ is directly integrable (Faraoni, 1999) and will be skipped henceforth. One can immediately see that by means of the change of function $u = \frac{a'}{a}$ the following Riccati equation is obtained:

$$u' + cu^2 + \kappa c = 0. \tag{5}$$

Employing now $u = \frac{1}{c}\frac{w'_\kappa}{w_\kappa}$ one gets the very simple second-order differential equation:

$$w''_\kappa + \kappa c^2 w_\kappa = 0. \tag{6}$$

For $\kappa = 1$ the solution of the latter is $w_1 = W_1 \cos(c\eta + d)$, where d is an arbitrary phase, implying $a_1(\eta) = A_1[\cos(c\eta + d)]^{1/c}$, whereas for $\kappa = -1$ one gets $w_{-1} = W_{-1}\sinh(c\eta)$ and therefore $a_{-1}(\eta) = A_{-1}\,[\sinh(c\eta)]^{1/c}$, where W_{+1} and A_{+1} are amplitude parameters. These are the same solutions as in the textbook procedures.

The point now is that the Riccati solution $u_p = \frac{1}{c}\frac{w'}{w}$ mentioned above is only the particular solution, i.e., $u_{p,1} = -\tan(c\eta)$ and $u_{p,-1} = \coth(c\eta)$ for $\kappa = \pm 1$, respectively. The particular Riccati solutions are closely related to the common factorizations of the second-order linear differential, equations that are directly related to the well-known Darboux isospectral transformations (Matveev and Salle, 1990). Indeed, Eq. (6) can be written

$$w'' - c(-\kappa c)\,w = 0 \tag{7}$$

and also in factorized form using Eq. (6) one gets ($D_\eta = \frac{d}{d\eta}$)

$$(D_\eta + cu_p)(D_\eta + cu_p)\, w = w'' - c\big(u'_p + cu_p^2\big)\, w = 0. \tag{8}$$

To fix the ideas, we shall call Eq. (8) the bosonic linear equation. On the other hand, the supersymmetric partner (or fermionic) equation of Eq. (8) will be

$$\begin{aligned}(D_\eta + cu_p)(D_\eta + cu_p)\, w_{\rm f} &= w''_{\rm f} - c\big(u'_p + cu_p^2\big) w_{\rm f}\\ &= w''_{\rm f} - c \cdot c_{\kappa,\rm susy} w_{\rm f} = 0,\end{aligned} \tag{9}$$

which is related to the fermionic Riccati equation

$$-u' + cu^2 - c_{\kappa,\rm susy} = 0. \tag{10}$$

Thus, one can write

$$c_{\kappa,\rm susy}(\eta) = -u'_p + cu_p^2 = \begin{cases} c(1 + 2\tan^2 c\eta) & \text{if } \kappa = 1\\ c(-1 + 2\coth^2\ c\eta) & \text{if } \kappa = 1\end{cases}$$

for the supersymmetric partner adiabatic index. The solutions $w_{\rm f}$ are $w_{\rm f} = \frac{c}{\cos(c\eta + d)}$ and $w_{\rm f} = \frac{c}{\sinh(c\eta)}$ for $\kappa = 1$ and $\kappa = -1$, respectively.

Introducing the (quantum momentum) operator $P_\eta = -i D_\eta$, we can write the fermionic equations as follows:

$$(-P_\eta - icu_p)(P_\eta - icu_p)\, w_{\rm f} = -P_\eta^2 w_{\rm f} - c\big(-iP_\eta u_p + cu_p^2\big)\, w_{\rm f}, \tag{11}$$

whereas the bosonic case is

$$(P_\eta - icu_p)(-P_\eta - icu_p)\, w_{\rm b} = -P_\eta^2 w_{\rm b} - c\big(iP_\eta u_p + cu_p^2\big)\, w_{\rm b}, \tag{12}$$

There is a more general factorization of the bosonic equation (Mielnik, 1984; Rosu, 1999)

$$(D_\eta + cu_{\rm g})(D_\eta - cu_{\rm g})\, w_{\rm g} = w''_{\rm g} - c(u'_{\rm g} + cu_{\rm g}^2)\, w_{\rm g} = w''_{\rm g} + \kappa c c(\eta;\lambda)\, w_{\rm g} = 0, \tag{13}$$

which is given in terms of the general Riccati solution $u_{\rm g}(\eta)$ of the fermionic Riccati Eq. (10)

$$u_{\rm g}(\eta;\lambda) = u_p(\eta) - \frac{1}{c} D_\eta[\ln(I_\kappa(\eta) + \lambda)] = D_\eta\left[\ln\left(\frac{w_\kappa(\eta)}{I_\kappa(\eta) + \lambda}\right)^{\frac{1}{c}}\right] \tag{14}$$

and yields the one-parameter family of effective adiabatic indices $c_\kappa(\eta;\lambda)$

$$\begin{aligned}-\kappa c_\kappa(\eta;\lambda) = cu_{\rm g}^2(\eta;\lambda) + \frac{du_{\rm g}(\eta;\lambda)}{d\eta} &= -\kappa c - \frac{2}{c} D_\eta^2[\ln(I_\kappa(\eta) + \lambda)]\\ &= -\kappa c - \frac{4w_\kappa(\eta)w'_\kappa(\eta)}{c(I_\kappa(\eta) + \lambda)} + \frac{2w_\kappa^4(\eta)}{c(I_\kappa(\eta) + \lambda)^2},\end{aligned} \tag{15}$$

where $I_\kappa(\eta) = \int_0^\eta w_\kappa^2(y)\,dy$, if we think of a half-line problem for which λ is a positive integration constant thereby considered as a free parameter of the method.

All $c_\kappa(\eta;\lambda)$ have the same supersymmetric partner index $c_{\kappa\,\mathrm{susy}}(\eta)$ obtained by deleting the appropriately normalized zero mode solution w_κ. They may be considered as intermediates between the initial constant index κc and the supersymmetric partner index $c_{\kappa,\mathrm{susy}}(\eta)$. From Eq. (13) one can infer the new parametric "zero mode" solutions of the universe for the family of barotropic indices $c_\kappa(\eta;\lambda)$ as follows:

$$w_{\mathrm{g}}(\eta;\lambda) = \frac{w_\kappa(\eta)}{I_\kappa(\eta)+\lambda} \Longrightarrow a_{\mathrm{g}}(\eta,\lambda) = \left(\frac{w_\kappa(\eta)}{I_\kappa(\eta)+\lambda}\right)^{\frac{1}{c}}. \tag{16}$$

Before closing this section, we recall an interesting point. Since what we have done here is to use the Darboux transformations at the level of cosmological evolutionary equations (i.e., equations of motion of the scale factor of the FRW cosmologies) a natural question is what is the effect of such transformations at the level of any Lagrangian of the cosmological fluid mechanics. The answer to this question has been already provided in the literature. Neto and Filho (1997) have shown that in general the application of the Darboux transformations is equivalent to the addition of a total time derivative of a purely imaginary function to the Lagrangian and later, Samsonov (1998) using the coherent state approach confirmed their result.

3. DIRAC-LIKE FORMALISM

The Dirac equation in the susy nonrelativistic formalism has been discussed by Cooper *et al.* in 1988 (Cooper *et al.*, 1988; Hughes *et al.*, 1986; Nogami and Toyama, 1998). They showed that the Dirac equation with a Lorentz scalar potential is associated with a susy pair of Schroedinger Hamiltonians. This result has been used later by many authors. In mathematical terms it is only a simple approach for matrix differential equations. Here we present an application to barotropic PRW cosmologies that we find not to be a trivial exercise except for the uncoupled 'zero-mass' case (see 3.1).

(3.1.) Let us introduce now the following two Pauli matrices $\alpha = -i\sigma_y = -\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $\beta = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and write a cosmological Dirac equation

$$H_D^{FRW} W = [i\sigma_y P_\eta + \sigma_x(icu_p)]\, W = 0, \tag{17}$$

where $W = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ is a two-component "zero-mass" spinor. This is equivalent to the following decoupled equations:

$$-P_\eta w_1 + icu_p w_2 = 0 \tag{18}$$

$$+P_\eta w_1 + icu_p w_2 = 0. \tag{19}$$

Solving these equations one gets $w_1 \propto 1/\cos(c\eta)$ and $w_2 \propto \cos(c\eta)$ for $\kappa = 1$ cosmologies and $w_1 \propto 1/\sinh(c\eta)$ and $w_2 \propto \sinh(c\eta)$ for $\kappa = -1$ cosmologies. Thus, we obtain

$$W = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} w_{\rm f} \\ w_{\rm b} \end{pmatrix}.$$

This shows that the matrix "zero-mass" Dirac equation is equivalent to the two linear second-order differential equations for the bosonic and fermionic cosmological components.

(3.2.) Consider now the following Dirac equation:

$$H_D^{FRW} W = [i\sigma_y P_\eta + \sigma_x(icu_p + K)]W = KW, \tag{20}$$

where K is a positive real constant. In the left-hand side, K stands as a mass parameter of the Dirac spinor, whereas on the right-hand side it corresponds to the energy parameter, i.e., $E = K$. Thus, we have a Dirac equation for a spinor of mass K at the fixed energy $E = K$. This equation can be written as the following system of coupled equations:

$$-P_\eta w_1 + (icu_p + K)\, w_1 = K w_2 \tag{21}$$

$$P_\eta w_2 + (icu_p + K)\, w_2 = K w_1. \tag{22}$$

This system is equivalent to the following second-order equations for the two spinor components, respectively:

$$-P_\eta^2 w_i - c\left[i(\mp P_\eta - 2K)\, u_p + cu_p^2\right] w_i = 0, \tag{23}$$

where the subindex $i = 1, 2$.

The fermionic spinor component can be found directly as solutions of

$$D_\eta^2 w_1^+ - \left[c^2(1 + 2\tan^2 c\eta) + 2icK \tan c\eta\right] w_1^+ = 0 \quad \text{for}\, \kappa = 1 \tag{24}$$

and

$$D_\eta^2 w_1^- - \left[c^2(-1 + 2\coth^2 c\eta) - 2icK \coth c\eta\right] w_1^- = 0 \quad \text{for}\, \kappa = -1, \tag{25}$$

whereas the bosonic components are solutions of

$$D_\eta^2 w_2^+ + \left[c^2 - 2icK \tan c\eta\right] w_2^+ = 0 \quad \text{for}\, \kappa = 1 \tag{26}$$

and

$$D_\eta^2 w_2^- + \left[-c^2 + 2icK \coth c\eta\right] w_2^- = 0 \quad \text{for}\, \kappa = -1. \tag{27}$$

The solutions of the bosonic equations are expressed in terms of the Gauss hypergeometric functions ${}_2F_1$ in the variables $y = e^{ic\eta}$ and $y = e^{c\eta}$, respectively

$$w_2^+ = Ay^{-p}{}_2F_1\left[-\frac{1}{2}(p+iq); -\frac{1}{2}(p-iq); 1-p; -y^2\right] + By^p{}_2F_1\left[\frac{1}{2}(p-iq); \frac{1}{2}(p+iq), 1+p; -y^2\right] \tag{28}$$

and

$$w_2^- = C(-1)^{-\frac{i}{2}r}y^{-ir}{}_2F_1\left[-\frac{i}{2}(r+is), -\frac{i}{2}(r-is), 1-ir; y^2\right] + D(-1)^{\frac{i}{2}r}y^{ir}{}_2F_1\left[\frac{i}{2}(r-s), -\frac{i}{2}(r+s), 1+ir; y^2\right], \tag{29}$$

respectively. The parameters are the following: $p = (-1 - \frac{2K}{c})^{\frac{1}{2}}$, $q = (1 - \frac{2K}{c})^{\frac{1}{2}}$, $r = (-1 - i\frac{2K}{c})^{\frac{1}{2}}$, $s = (-1 + i\frac{2K}{c})^{\frac{1}{2}}$, whereas A, B, C, D are superposition constants.

It is not necessary to try to find the general fermionic solutions through the analysis of their differential equations (24) and (25) because they are related in a known way to the bosonic solutions (Boya *et al.*, 1998). The general fermionic solutions can be obtained easily if one argues that the particular fermionic zero mode is the inverse of a particular bosonic zero mode and constructing the other independent zero mode solution as in textbooks. Thus

$$w_1^{\pm} = \frac{1 + k\int^y \left[w_2^{\pm}\right]^2 dz}{w_2^{\pm}}, \tag{30}$$

where k is an arbitrary constant.

(3.3.) The most general case in this scheme is to consider the following matrix Dirac-like equation:

$$\left[i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} P_\eta + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} icu_p + K_1 & 0 \\ 0 & icu_{\rm g} + K_2 \end{pmatrix}\right]\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}. \tag{31}$$

Proceeding as in 3.2 one finds the coupled system of first-order differential equations

$$[P_\eta + icu_{\rm g} + K_2]\, w_2 = K_1 w_1 \tag{32}$$

$$[-P_\eta + icu_p + K_1]\, w_1 = K_2 w_2 \tag{33}$$

and the equivalent second-order differential equations

$$-P_\eta{}^2 w_i + [ic(u_p - u_{\rm g}) + (K_1 - K_2)]P_\eta w_i + [ic(\pm P_\eta u_i + K_1 u_{\rm g} + k_2 u_p) - c^2 u_p u_{\rm g}]\, w_i = 0, \tag{34}$$

where the subindex $i = 1, 2$, and u_1 and u_2 correspond to u_p and $u_{\rm g}$, respectively. In the D_η notation this equation reads

$$D_\eta^2 w_i + [c\Delta u_{pg} - i\Delta K)]\, D_\eta w_i + [c(\pm D_\eta u_i + (iK_1 u_{\rm g} + K_2 u_p)) - c^2 u_p u_{\rm g}]w_i = 0. \tag{35}$$

Under the gauge transformation

$$w_i = z_i \exp\left(-\frac{1}{2}\int^\eta [c\Delta u_{pg} - i\Delta K]\, d\tau\right) = z_i(\eta)\frac{e^{\frac{1}{2}i\eta\Delta K}}{(I_\kappa + \lambda)^{\frac{1}{2}}} \tag{36}$$

one gets

$$-P_\eta^2 z_i + Q_i(\eta) z_i = 0, \quad D_\eta^2 z_i + Q_i(\eta)\, z_i = 0, \tag{37}$$

where

$$Q_i(\eta) = [c(\pm D_\eta u_i + (iK_1 u_{\rm g} + K_2 u_p)) - c^2 u_p u_{\rm g}] - \frac{1}{2}D_\eta[c\Delta u_{pg}] - \frac{1}{4}[c\Delta u_{pg} - i\Delta K]^2 \tag{38}$$

for $i = 1, 2$ respectively. Q_i are complicated effective barotropic indices and we were not able to find analytical solutions of Eq. (37).

The corresponding Dirac spinor is of the following form

$$W(\lambda, K_1, K_2) = \begin{pmatrix} z_1(K_1) \\ z_2(\lambda, K_2) \end{pmatrix} = \begin{pmatrix} w_{\rm f}(K_1) \\ w_{\rm g}(\lambda, K_2) \end{pmatrix},$$

where $w_{\rm g}(\lambda, K_2)$ is given by Eq. (16) for $K_1 = K_2 = 0$. For $\lambda \to \infty$ one obtains $W(\lambda, 0, 0) \to W$. In addition, $u_{\rm g} \to u_p$ and for $K_1 = K_2 = K$ one gets the particular case in 3.2.

4. CONCLUSIONS

We come now to the interpretation of the mathematical results that we displayed in the previous sections. An examination of the formulas (23–26) and (37) show that the parameters K introduce an imaginary part in the effective adiabatic index of the cosmological fluid. Thus, the supersymmetric techniques presented in this research letter are a particular way to consider dissipation and instabilities in the ideal case of barotropic FRW cosmologies. More general scale factors of barotropic FRW universes incorporating a well-defined type of dissipation can be obtained from the "zero-modes" $w^{\pm}_{1,2}$ by means of the relation $a \sim w^{1/c(\eta;K)}$. The indices $c(\eta; K)$ are redefined (effective) adiabatic indices that can be inferred from the formulas (23–26) and (37), respectively.

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